Finite Math - J-term 2017

Lecture Notes - 1/17/2017

Homework

- Section 4.3 - 41, 42, 52, 54, 59, 76
- Section 4.4 - 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 23, 27, 30, 31, 32, 35, 37, 42, 44, 55, 57, 58

Section 4.3 - Gauss-Jordan Elimination

Example 1. *Solve by Gauss-Jordan elimination:*

$$\begin{array}{rcrcrcr} 2x & - & y & - & 3z & = & 8 \\ x & - & 2y & & & = & 7 \end{array}$$

Solution. *The augmented matrix is*

$$\left[\begin{array}{ccc:c} 2 & -1 & -3 & 8 \\ 1 & -2 & 0 & 7 \end{array}\right]$$

Begin as always, by getting the 1 *in the top left*

$$\left[\begin{array}{ccc:c} 2 & -1 & -3 & 8 \\ 1 & -2 & 0 & 7 \end{array}\right] \overset{R_1 \leftrightarrow R_2}{\sim} \left[\begin{array}{ccc:c} 1 & -2 & 0 & 7 \\ 2 & -1 & -3 & 8 \end{array}\right]$$

Then getting the zero below it

$$\left[\begin{array}{ccc:c} 1 & -2 & 0 & 7 \\ 2 & -1 & -3 & 8 \end{array}\right] \overset{R_2 - 2R_1 \to R_2}{\sim} \left[\begin{array}{ccc:c} 1 & -2 & 0 & 7 \\ 0 & 3 & -3 & -6 \end{array}\right]$$

Now we get the 1 *in the second column*

$$\left[\begin{array}{ccc:c} 1 & -2 & 0 & 7 \\ 0 & 3 & -3 & -6 \end{array}\right] \overset{\frac{1}{3}R_2 \to R_2}{\sim} \left[\begin{array}{ccc:c} 1 & -2 & 0 & 7 \\ 0 & 1 & -1 & -2 \end{array}\right]$$

then use this to get a zero above it

$$\left[\begin{array}{ccc:c} 1 & -2 & 0 & 7 \\ 0 & 1 & -1 & -2 \end{array}\right] \overset{R1 + 2R_2 \to R_1}{\sim} \left[\begin{array}{ccc:c} 1 & 0 & -2 & 3 \\ 0 & 1 & -1 & -2 \end{array}\right]$$

This tells us that $x - 2z = 3$ *and* $y - z = -2$*. Since* z *is in both equations, we will let* $z = t$*, then we have* $x = 2t + 3$ *and* $y = t - 2$*. So the solutions is*

$$x = 2t + 3, y = t - 2, z = t$$

for real numbers t*.*

Example 2. *Solve by Gauss-Jordan elimination:*

$$\begin{array}{rcrcrcr} 2x_1 & + & 4x_2 & - & 6x_3 & = & 10 \\ 3x_1 & + & 3x_2 & - & 3x_3 & = & 6 \end{array}$$

Solution. $x_1 = -t - 1, x_2 = 2t + 3, x_3 = t$

Example 3. *A company that rents small moving trucks wants to purchase* 16 *trucks with a combined capacity of* 19,200 *cubic feet. Three different types of trucks are available: a cargo van with a capacity of* 300 *cubic feet, a* 15*-foot truck with a capacity of* 900 *cubic feet, and a* 24*-foot truck with a capacity of* 1,500*-cubic feet. How man of each type should the company purchase?*

Solution. $t-8$ *cargo vans,* $-2t+24$ *of the* 15*-foot trucks, and* t *of the* 24 *foot trucks, where* $t = 8$, 9, 10, 11, *or* 12

Section 4.4 - Matrices: Basic Operations

Addition and Subtraction. First, let's define what it means for two matrices to be equal.

Definition 1 (Equal)**.** *Two matrices are* equal *if they are the same size and the corresponding elements in each matrix are equal.*

For example, the equality

$$\begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix} = \begin{bmatrix} u & v \\ w & x \\ y & z \end{bmatrix}$$

is true if and only if

$$\begin{array}{ll} a = u & b = v \\ c = w & d = x \\ e = y & f = z \end{array}.$$

In order to add or subtract matrices **they must be the same size**.

- When adding matrices, we just add the corresponding elements.
- When subtracting matrices, we just subtract the corresponding elements.

Example 4. *Find the indicated operations*

(a)

$$\begin{bmatrix} 3 & 2 \\ -1 & -1 \\ 0 & 3 \end{bmatrix} + \begin{bmatrix} -2 & 3 \\ 1 & -1 \\ 2 & -2 \end{bmatrix}$$

(b)

$$\begin{bmatrix} 3 & 2 \\ 5 & 0 \end{bmatrix} - \begin{bmatrix} 2 & -2 \\ 3 & 4 \end{bmatrix}$$

(c)

$$\begin{bmatrix} 3 & 2 & -1 \\ -1 & 0 & 3 \end{bmatrix} + \begin{bmatrix} -2 & 3 \\ 1 & -1 \\ 2 & -2 \end{bmatrix}$$

Solution.

(a)

$$\begin{bmatrix} 3 & 2 \\ -1 & -1 \\ 0 & 3 \end{bmatrix} + \begin{bmatrix} -2 & 3 \\ 1 & -1 \\ 2 & -2 \end{bmatrix} = \begin{bmatrix} 3+(-2) & 2+3 \\ -1+1 & -1+(-1) \\ 0+2 & 3+(-2) \end{bmatrix} = \begin{bmatrix} 1 & 5 \\ 0 & -2 \\ 2 & 1 \end{bmatrix}$$

(b)

$$\begin{bmatrix} 3 & 2 \\ 5 & 0 \end{bmatrix} - \begin{bmatrix} 2 & -2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 3-2 & 2-(-2) \\ 5-3 & 0-4 \end{bmatrix} = \begin{bmatrix} 1 & 4 \\ 2 & -4 \end{bmatrix}$$

(c) *These matrices are not the same size and so cannot be added.*

Example 5. *Find the indicated operations*

(a)

$$\begin{bmatrix} 2 & -3 \\ 1 & 2 \end{bmatrix} + \begin{bmatrix} 1 & -1 \\ 0 & -2 \end{bmatrix}$$

(b)

$$\begin{bmatrix} -3 & 2 \\ 4 & -1 \end{bmatrix} - \begin{bmatrix} -2 & 5 \\ -1 & 3 \end{bmatrix}$$

(c)

$$\begin{bmatrix} 3 \\ -1 \\ 3 \end{bmatrix} + \begin{bmatrix} -2 & 3 & -2 \end{bmatrix}$$

Scalar Multiplication. If k is a number and M is a matrix, we can form the scalar product kM by just multiplying every element of M by k.

Example 6. *Find*

$$-2\begin{bmatrix} 3 & -1 & 0 \\ -2 & 1 & 3 \\ 0 & -1 & -2 \end{bmatrix}$$

Solution.

$$-2\begin{bmatrix} 3 & -1 & 0 \\ -2 & 1 & 3 \\ 0 & -1 & -2 \end{bmatrix} = \begin{bmatrix} -2(3) & -2(-1) & -2(0) \\ -2(-2) & -2(1) & -2(3) \\ -2(0) & -2(-1) & -2(-2) \end{bmatrix} = \begin{bmatrix} -6 & 2 & 0 \\ 4 & -2 & -6 \\ 0 & 2 & 4 \end{bmatrix}$$

Example 7. *Find*

$$5\begin{bmatrix} 1 & -1 \\ 0 & -2 \\ 2 & -3 \\ 3 & 3 \end{bmatrix}$$

Matrix Multiplication. In order to define matrix multiplication, it is easier to first define the product of a row matrix with a column matrix.

Definition 2. *Suppose we have a $1 \times n$ row matrix A and an $n \times 1$ column matrix B where*

$$A = \begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix} \quad \textit{and} \quad B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}.$$

Then

$$AB = \begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n.$$

It is very important that the number of columns in A matches the number of rows in B.

Example 8. *Find*

$$\begin{bmatrix} -1 & 0 & 3 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ 4 \\ -1 \end{bmatrix}$$

Solution.

$$\begin{bmatrix} -1 & 0 & 3 & 2 \end{bmatrix}\begin{bmatrix} 2 \\ 3 \\ 4 \\ -1 \end{bmatrix} = (-1)(2)+(0)(3)+(3)(4)+(2)(-1) = -2+0+12-2 = 8$$

Example 9. *Find*

$$\begin{bmatrix} 2 & -1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}$$

Definition 3 (Matrix Multiplication)**.** *Let A be an $m \times p$ matrix and let B be a $p \times n$ matrix. Let R_i denote the matrix formed by the i^{th} row of A and let C_j denote the matrix formed by the j^{th} column of B. Then the ij^{th} element of the matrix product AB is R_iC_j.*

Remark 1. *It is very important that the number of columns of A matches the number of rows of B, otherwise the products R_iC_j would not be able to be defined. That is, if A is an $m \times n$ matrix and B is an $p \times q$ matrix, the product AB is defined if and only if $n = p$.*

Example 10. *Let* $A = \begin{bmatrix} -1 & 0 & 3 & -2 \\ 1 & 2 & 2 & 0 \end{bmatrix}$, $B = \begin{bmatrix} -1 & 1 \\ 2 & 3 \\ 1 & 0 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 2 \\ -1 & -2 \end{bmatrix}$, $D = \begin{bmatrix} -2 & 4 \\ 1 & -2 \end{bmatrix}$. *Find the following products, if possible.*

(a) AB

(b) BA

(c) CD

(d) DC

(e) CB

(f) D^2

Solution.

(a) *Since A is 2×4 and B is 3×2, the product AB is not defined.*

(b)

$$\begin{aligned}
BA &= \begin{bmatrix} -1 & 1 \\ 2 & 3 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} -1 & 0 & 3 & -2 \\ 1 & 2 & 2 & 0 \end{bmatrix} = \begin{bmatrix} R_1C_1 & R_1C_2 & R_1C_3 & R_1C_4 \\ R_2C_1 & R_2C_2 & R_2C_3 & R_2C_4 \\ R_3C_1 & R_3C_2 & R_3C_3 & R_3C_4 \end{bmatrix} \\
&= \begin{bmatrix} \begin{bmatrix} -1 & 1 \end{bmatrix}\begin{bmatrix} -1 \\ 1 \end{bmatrix} & \begin{bmatrix} -1 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 2 \end{bmatrix} & \begin{bmatrix} -1 & 1 \end{bmatrix}\begin{bmatrix} 3 \\ 2 \end{bmatrix} & \begin{bmatrix} -1 & 1 \end{bmatrix}\begin{bmatrix} -2 \\ 0 \end{bmatrix} \\
\begin{bmatrix} 2 & 3 \end{bmatrix}\begin{bmatrix} -1 \\ 1 \end{bmatrix} & \begin{bmatrix} 2 & 3 \end{bmatrix}\begin{bmatrix} 0 \\ 2 \end{bmatrix} & \begin{bmatrix} 2 & 3 \end{bmatrix}\begin{bmatrix} 3 \\ 2 \end{bmatrix} & \begin{bmatrix} 2 & 3 \end{bmatrix}\begin{bmatrix} -2 \\ 0 \end{bmatrix} \\
\begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} -1 \\ 1 \end{bmatrix} & \begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 2 \end{bmatrix} & \begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} 3 \\ 2 \end{bmatrix} & \begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} -2 \\ 0 \end{bmatrix} \end{bmatrix} \\
&= \begin{bmatrix} (-1)(-1)+(1)(1) & (-1)(0)+(1)(2) & (-1)(3)+(1)(2) & (-1)(-2)+(1)(0) \\ (2)(-1)+(3)(1) & (2)(0)+(3)(2) & (2)(3)+(3)(2) & (2)(-2)+(3)(0) \\ (1)(-1)+(0)(1) & (1)(0)+(0)(2) & (1)(3)+(0)(2) & (1)(-2)+(0)(0) \end{bmatrix} \\
&= \begin{bmatrix} 2 & 2 & -1 & 2 \\ 1 & 6 & 12 & -4 \\ -1 & 0 & 3 & -2 \end{bmatrix}
\end{aligned}$$

(c)

$$\begin{aligned}
CD &= \begin{bmatrix} 1 & 2 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} -2 & 4 \\ 1 & -2 \end{bmatrix} \\
&= \begin{bmatrix} (1)(-2)+(2)(1) & (1)(4)+(2)(-2) \\ (-1)(-2)+(-2)(1) & (-1)(4)+(-2)(-2) \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}
\end{aligned}$$

Solution. (d) $\begin{bmatrix} -6 & -12 \\ 3 & 6 \end{bmatrix}$

(e) *Not defined.*

(f) $\begin{bmatrix} 8 & -16 \\ -4 & 8 \end{bmatrix}$

Remark 2. *Note that parts (c) and (d) show that matrix multiplication is not commutative. That is, it is not necessarily true that $AB = BA$ for matrices A and B, even if both matrix products are defined.*